Adaptive Arrival Price

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Outline

- Evolution of algorithmic trading
- “Classic” arrival price algorithms
- Price adaptive algorithms
  - Single-update and multi-update strategies
  - Dynamic programming
  - Improved shortfall statistics and efficient frontiers
- Extensions
  - Bayesian adaptive trading with price trend
Electronic/Algorithmic Trading

- Use computers to execute orders
- Agency trading
  - Executing orders for clients
  - Investment decision is made
- Large and increasing fraction of total order flow
  - 92% of hedge funds and
    11% of all trades, 17% by 2007 (Tabb Group)
  - Expected 50% of traded volume by 2010 (The Economist, ‘07)
Evolution of Algorithmic Trading

Use computers to execute orders (agency trading)

1. VWAP
   - Automatization of routine trading tasks

2. Arrival Price
   - Quantitative modeling and optimization
   - Market impact, volatility, alpha, etc

3. Adaptivity
   - Execution trajectory responds to market behavior in a variety of ways
   - How to do this optimally?
**VWAP**

- Easy to understand and implement
  - “Spread trades out over time”
- Criticism
  - For large trades in illiquid securities, VWAP essentially reflects trade itself and provides little incentive for low-cost execution
  - Artificial: does not correspond to any investment goal
Arrival Price

**Benchmark:** Pre-trade (decision) price

If you could execute entire order instantly at this price, you would

**Why trade slowly?**
- Reduce market impact

**Why trade rapidly?**
- Minimize risk
- Anticipated drift

- “Trader‘s Dilemma“

- Control statistical properties of shortfall
  - Risk-Reward tradeoff
Arrival Price: Efficient Strategies

Mean-Variance (Markowitz optimality): Strategies that minimize

- $E$ for fixed $V$,
- $V$ for fixed $E$, or
- $E + \lambda V$ for risk aversion parameter $\lambda$

1. Minimal variance
2. Minimal expected cost

Equivalent formulations!
Arrival Price: Almgren/Chriss Trajectories

[Almgren, Chriss ’00]: **Static trajectories** specified at \( t=0 \)

- Efficient trajectories given by

\[
x(t) = \frac{\sinh(\kappa(T - t))}{\sinh(\kappa T)} X
\]

\( x(t) = \) stock holding at time \( t \)

\( T = \) time horizon

\( X = \) initial shares

Urgency \( \kappa \geq 0 \) controls curvature
Adaptivity

Intuitively, adaptivity makes a lot of sense

1. Adapt to varying volume and volatility
   - trade more when liquidity is present
   - trade less when volatility risk is lower

2. Adapt to price movement: “scaling”
   - trade faster or slower when price move is favorable?
Scaling in response to price motions

Common wisdom: Depending on asset price process

- Mean-reversion \(\implies\) aggressive in the money (AIM)
- Momentum \(\implies\) passive in the money (PIM)
- Pure random walk \(\implies\) no response

[Almgren, L.]: AIM improves mean-variance tradeoff, especially

- for nonzero risk aversion (middle of frontier)
- and large portfolio sizes

1. **Single-Update**: [Algorithmic Trading III, Spring 2007]
   - Adapt strategy exactly once during trading

2. **Multi-Update**: [In preparation]
   - Any desired degree of precision
Intuition behind AIM for Arrival Price

- Introduce **intertemporal anti-correlation** between
  - investment gains/losses in first part of execution, and
  - trading costs (market impact) in second part of execution
- If make money in first part, spend parts on higher impact
- Higher impact = trade faster (i.e. reduces market risk)
- Trade this volatility reduction for expected cost reduction
  (mean-variance tradeoff!)
- Caveat: “Cap winners, let losers run” is deadly if real price process has momentum
1. Single-Update

[Almgren, L. 2007]

- Follow Almgren-Chriss trajectory with urgency $\kappa_0$ until $T_*$.  
- At $T_*$ switch to different urgency $\kappa_1, \ldots, \kappa_n$ depending on performance up to $T_*$.  

If price up at $T_*$, trade faster in remainder.
Single-Update (cont.)

- Switch to \( \kappa_i \) if based on accumulated gain/loss at \( T_* \)

- \( C_0, C_1, \ldots, C_n \) cost on first and second part

\[
C_0 = \sigma \int_0^{T_*} x_0(t) \, dB(t) + \eta \int_0^{T_*} x_0'(t)^2 \, dt \sim \mathcal{N}(E_0, V_0)
\]

\[
C_j = \sigma \int_{T_*}^{T} x_j(t) \, dB(t) + \eta \int_{T_*}^{T} x_j'(t)^2 \, dt \sim \mathcal{N}(E_j, V_j)
\]

- Explicit expressions for \( E[C_i] \) and \( \text{Var}[C_i] \)

- Explicit expression for \( E \) and \( \text{Var} \) of composite strategy
  \( \rightarrow \) Numerical optimization of criterion \( E + \lambda \text{Var} \) over \( \kappa_0, \ldots, \kappa_n \)
Single-Update: Numerical Results

Family of frontiers parametrized by market power $\mu$

- $\mu = 0$
- $\mu > 0$ (i.e. $X \gg 0$)

Larger relative improvement for large portfolios $\mu \gg 0$

AC static frontiers coincide with $\mu \to 0$
2. Multiple Updates: Dynamic Programming

- Single-update does NOT generalize to multiple updates
- “E + \lambda \text{Var}“ not amenable to dynamic programming
  - Squared expectation in \text{Var}[X] = \text{E}[X^2] - \text{E}[X]^2

[Almgren, L. 2008]: Define value function

\[ J_k(x, c) = \text{“minimal variance for k periods and } x \text{ shares s.t. expected cost at most } c” \]

\( J_{k-1}(x, c) \) and optimal strategies for k-1 periods

Optimal Markovian one-step control

\( J_k(x, c) \) and optimal strategies for k periods

Markov property for mean-variance efficient strategies
Dynamic Programming (cont.)

We want to determine $J_k(x, c)$

Situation:
- **Sell program** (buy program analogously)
- $k$ periods and $x$ shares left
- limit for expected cost is $c$
- current stock price $S$
- next price innovation is $\sigma \xi \sim N(0, \sigma^2)$

Construct optimal strategy $\pi$ for $k$ periods

1. **In current period** sell $y$ shares at $\tilde{S} = S - h(y)$
2. Use an efficient strategy $\pi'$ for remaining $x - y$ shares in remaining $k-1$ periods
Dynamic Programming (cont.)

Note: \( y \) must be deterministic, but when we begin \( \pi' \), outcome of \( \xi \) is known, i.e. we may choose \( \pi' = \pi'(\xi) \) depending on \( \xi \)

\[ \Rightarrow \text{Specify } \pi'(\xi) \text{ by its expected cost } z(\xi) \]
\[ \Rightarrow \text{Strategy } \pi \text{ defined by } y \in \mathbb{R} \text{ and control function } z(\xi) \]

- Expressions for cost of strategy \( \pi \) conditional on \( \xi \)
  - \( \mathbb{E}[C(\pi) | \xi] \) and \( Var[C(\pi) | \xi] \)
  - Use the laws of total expectation and variance
    - \( \mathbb{E}[C(\pi)] = \mathbb{E}[\mathbb{E}[C(\pi) | \xi]] \)
    - \( Var[C(\pi)] = \mathbb{E}[Var[C(\pi) | \xi]] + Var[\mathbb{E}[C(\pi) | \xi]] \)

Optimization of \((\mathbb{E}[C], Var[C])\) by means of \( y \) and \( z(\xi) \)
Dynamic Programming (cont.)

Value function recursion:

\[
J_k(x, c) = \min_{(y, z) \in G_k(x, c)} \left\{ \text{Var}[z(\xi) - \sigma \xi (x-y)] + E[J_{k-1}((x-y), z(\xi))] \right\}
\]

where

\[
G_k(x, c) = \left\{ (y, z) \left| \begin{array}{l}
0 \leq y \leq x, \quad E[z(\xi)] + \eta y^2 \leq c \\
y \in \mathbb{R}, \quad z \in L^1(\Omega; \mathbb{R})
\end{array} \right. \right\}
\]

(For linear price impact)
Solving the Dynamic Program

- Series of one-period optimization problems
- Each step: Determine optimal control
  - #shares to sell in next period
  - Target expected cost for remainder as function $z(\xi)$ of next period stock price change
- No closed form solution $\rightarrow$ numerical approximation
- Nice property: Convex constrained problems
Behavior of Adaptive Strategy

Aggressive in the Money (AIM)

Formally: Control function $z(\zeta)$ is monotone increasing

Example: sell program

Spend windfall gains on increased impact costs to reduce total variance

“If price goes up, sell faster“
Efficient Frontiers

Sample cost PDFs:

Similar results as for single-update

- Larger relative improvement for large portfolios
- Market power $\mu$

\[ \frac{\text{Var}[C]}{V_{\text{lin}}} \]

\[ \frac{E[C]}{E_{\text{lin}}} \]

$N = 50$

$\mu$ increasing
Single Update vs. Multi-Update

- Full improvement by multi-update
- Significant improvement even by single-update
- Multi-update naturally more computational intensive
- Single-update offers good value for low computation
Other Criteria

- Instead of mean-variance tradeoff: Utility functions
  - Exponential utility (CARA): \( u(y) = -\exp(-\alpha y) \)
  - Power law utility: \( u(y) = \frac{y^{1-\gamma} - 1}{1 - \gamma} \)
  - etc.

- Optimal strategies are AIM or PIM depending on utility
  [Schied, Schöneborn ’08]

- Advantage of mean-variance optimization
  - Clear and intuitive meaning
  - Corresponds to how trade results are reported in practice
  - Independent of client’s wealth
Extensions

- Non-linear impact functions and cost models
- Multiple securities (program trading)
- Other asset price processes
  - Price momentum (drift)
  - Mean-Reversion
  - Non-Gaussian returns
- etc.
Trading with Price Trend

”Daily Trading Cycle”: institutional traders make decisions overnight and trade across the following day

- Price momentum, if large net positions being traded

Market Model

- Stock price: Brownian motion with unknown drift $\alpha$

$$S(t) = S_0 + \alpha t + \sigma B(t) \quad \text{with} \quad \alpha \sim \mathcal{N}(\bar{\alpha}, \nu^2)$$

- Prior estimate for drift, update this belief using price observations
- Temporary and permanent market impact

Optimal risk-neutral strategy (buy program)

\[ x(t) = \text{trading trajectory (shares remaining) in continuous time} \]
\[ v(t) = \text{instantaneous trading rate} \]

\[ [\text{Almgren, L. 2007}]: \text{Optimal dynamic strategy given by instantaneous trade rate} \]

\[ v_\ast(t) = \frac{x(t)}{T - t} + \hat{\alpha}(t, S(t)) \cdot \frac{T - t}{4\eta} \]

- \( \hat{\alpha}(t, S(t)) \): Best estimate of drift using prior and \( S(t) \)
- Trade rate of re-computed static trajectory with current best drift estimate
  - Locally optimal myopic strategy = Global optimal solution
  - Highly dynamic
Trading with Price Trend: Examples

Buy-program with significant upwards drift prior

Prior drift estimate

Posterior drift estimate

Stock price path

Slow down trading, if cost lower than expected (PIM)
Conclusion

- Adaptivity is the new frontier of algorithmic trading
- Single-update and multi-update arrival price algorithms
  - Pure random walk, no momentum or reversion
  - Straightforward mean-variance criterion
  - Dynamic programming approach
  - Strategies are “aggressive-in-the-money“
- Substantially *better than static Almgren/Chriss trajectories*
  - Improved efficient frontiers
  - Relative improvement bigger for large portfolios
  - New market power parameter $\mu$
Thank you very much for your attention!

Questions?